Ordinary Differential Equation Notes

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last updated 2024-01-12

1 First Order

Definition: A differential equation is an equation with an unknown function and it's derivative(s)

Example 1.0.1: $D_x y = 2x + y$
Solution: $y = \int 2x + 7dx = x^2 + 7x + C$

Example 1.0.2: $D_x y + y = 7$
Solution here is a little more complicated

1.1 Linear First Order

Definition: A linear first order differential equation is one such that it can be written $\frac{dy}{dx} + P(x)y = Q(x)$

Example 1.1.1: $D_x y = y(1-y) = y - y^2$ (the logistic model)
Solution: $D_x y = \frac{dy}{dx} = y(1-y) \Leftrightarrow \frac{dy}{y(1-y)} = dx \Leftrightarrow \int \frac{dy}{y(1-y)} = \int dx \Leftrightarrow u = 1 - \frac{1}{y} \Rightarrow -\int \frac{1}{u}du = x + C \Leftrightarrow -\ln|u| = -\ln|1-\frac{1}{y}| = x + C \Leftrightarrow \ln|1-\frac{1}{y}| = -x + C \Leftrightarrow$ Assuming $1 - \frac{1}{y} \geq 0,\ 1 - \frac{1}{y} = e^{-x+C} \Leftrightarrow -1 = e^{-x+C}y - y = y(e^{-x+C} - 1) \Leftrightarrow y = \frac{-1}{e^{-x+C}-1}$

Solving first order linear D.E.s The simplest general method for solving first order linear D.E.s ($\frac{dy}{dx} + P(x)y = Q(x)$) is to add an additional function $\mu(x)$:

$$D_x y + P(x)y = Q(x)$$
$$\Leftrightarrow \mu(x)(D_x y + P(x)y) = D_x(\mu(x)y)$$
$$\Leftrightarrow \mu(x)D_x y + \mu(x)P(x)y = \mu(x)D_x y + D_x\mu(x)y$$
$$\Leftrightarrow \mu(x)P(x) = D_x\mu(x)y$$
$$P(x) = \frac{\mu_x(x)}{\mu(x)} = \frac{d\mu}{\mu} \Rightarrow \int P(x)dx = \int \frac{d\mu}{\mu} = \ln\mu \Rightarrow \mu(x) = e^{\int P(x)dx}$$

And then $\mu(x)$ can be plugged back in to find y.

Theorem: If P,Q are continuous on an open interval, I, containing x_0, then the initial value problem (IVP) $\frac{dy}{dx} + P(x)y = Q(x)$, $y(x_0) = y_0$ has a unique solution $y(x)$ on I given by $y(x) = e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)dx + C\right)$ for an appropriate C

1.2 Substitution

If you have $\frac{dy}{dx} = f(x,y)$, substitute part of the e.q. with $v = \alpha(x,y)$, and then plug back into the e.q. at the end.

Usually you want to get a linear D.E. relative to v and $D_x v$ (for example $D_x v + P(x)v = Q(x)$) which is easier to solve.

Note 1: sometimes a second order D.E. can be reduced into a first order by substituting $v = q(D_x y) \Rightarrow D_x v = w(D_x^2 y)$. (e.g. $xD_x^2 y + D_x y = Q(x)$ sub $v = D_x y \Rightarrow D_x v = D_x^2 y \Rightarrow xD_x v + v = Q(x)$ is 1st order)

Node 2: you can substitute implicitly (e.g. $yD_x y + (D_x^2 y)^2 = 0$ sub $v = yD_x y \Rightarrow D_x v = (D_x y)^2 + yD_x^2 y \Rightarrow D_x v = 0 \Rightarrow yD_x y = y\frac{dy}{dx} = v = c \Rightarrow \int ydy = \int cdx + C$)

1.2.1 Special cases of substitution

homogenious equation: An equation of the type: $D_x y = F(\frac{y}{x}) \Rightarrow$ substitute $v = \frac{y}{x} \Rightarrow y = vx \Rightarrow D_x y = D_x(v)x + v$ which can be solved by $v`x + v = F(v) \Rightarrow \frac{D_x v}{F(v) - v} = \frac{1}{x}$

In general $f(x,y)dy = g(x,y)dx$ substitute $y = ux \Rightarrow \frac{dx}{x} = h(u)du \Rightarrow$ integrate.

Bernoulli euquation: An equation of the type: $D_x y + P(x)y = Q(x)y^n$ $\Rightarrow$ substitute $v = y^{1-n} \Rightarrow D_x v = (1-n)y^{-n}D_x y \Rightarrow D_x v + (1-n)P(x)v = (1-n)Q(x)$

Remember that n can be negative (for example $D_x y + P(x)y = Q(x)\frac{1}{y}$)

1.3 Exact equation

An equation $I(x,y) + J(x,y)D_x y = 0 \Leftrightarrow I(x,y)dx + J(x,y)dy = 0$ is exact iff $I_y = J_x$. Then $\exists\psi(x,y)$ s.t.

$$\psi_x(x,y) = I(x,y)$$
$$\psi_y(x,y) = J(x,y)$$

and $\psi(x,y) = c$ is a solution.

For an IVP, given $f(a) = b$, plugin $x = a, y = b$ into ψ, and solve for c.

1.3.1 Autonomous equation

An autonomous DE is one such that the independent variable (e.g. x, t) is not in the eq. (e.g. $D_x y = P(y)$).

An autonomous equation is always seperable

1.4 Seperable equation

A seperable equation is one of the form $D_x y = f(x)g(y)$. A seperable equation can be solved as follows: $D_x y = \frac{dy}{dx} = f(x)g(y) \Rightarrow \int \frac{dy}{g(y)} = \int f(x)dx + C$.

2 Second Order

A second order differential equation is a differential equation that includes the second derivative.

2.1 Second Order Constant Coefficient Homogeneous

A second order constant coefficient homogeneous D.E. is one with the form $aD_x^2 y + bD_x y + cy = 0$.

Theorem (Super Position): If a second order homogeneous D.E. has 2 solutions a, b, then $a + b$ is also a solution.

To solve a second order linear constant coefficient homogeneous differential equation $aD_x^2 y + bD_x y + cy = 0$ let $y = e^{rx}$. Then by plugging in we get:

$$\begin{aligned} &a(r^2 e^{rx}) + b(re^{rx}) + ce^{rx} = 0 \\ &\Leftrightarrow e^{rx}(ar^2 + br + c) = 0 \\ &\Leftrightarrow ar^2 + br + c = 0 \end{aligned}$$

which can be solved for $r = r_1, r_2 \Rightarrow y = Ae^{rx} + Be^{rx}\ \forall A, B$ by super position. If there is a repeated root ($r_1 = r_2$), than let $y = Ae^{rx} + Bxe^{rx}$, plug in, and solve.

2.2 Second Order Non-Homogeneous

Let $D_x^2 y + p(x)D_x y + q(x)y = g(x)$ be the 2nd order non-homogeneous D.E. For simplicity, let $L[y] = D_x^2 y + p(x)D_x y + q(x)y$ (so the D.E. is $L[y] = g(x)$). Then the solution is of the form $y = (c_1 y_1 + c_2 y_2 = y_h) + y_p$ where y_h is the solution to $L[y] = 0$. To find y_p there is really two methods:

2.2.1 Undetermined Coefficients

Refer to the following table to find the general equation for y_p based on $g(x)$:

$g(x)$	y_p
$P_n(x)$	$t^s Q_n(x)$
$P_n(x)e^{\alpha x}$	$t^s Q_n(x)e^{\alpha x}$
$P_n(x)e^{\alpha x}(\sin\beta t + \cos\beta t)$	$t^s e^{\alpha x}(Q_n(x)\sin\beta x + R_n(x)\cos\beta x)$

Where s is the smallest integer ≥ 0 s.t. y_p is not a solution to $L[y] = 0$ and $P_n(x), Q_n(x), R_n(x)$ are polynomials of degree n.

Then plug into $L[y_p] = g(t)$ and solve for the coefficients.

2.2.2 Variation of Parameters

Given $y_h = c_1y_1 + c_2y_2$ we wantto find u_1, u_2 s.t. $D_x(u_1)y_1 + D_x(u_2)y_2 = 0$ and $D_x(u_1)D_x(y_1) + D_x(u_2)D_x(y_2) = g(x)$. We can find precise values using the Wronskian.

Definition: The Wronskian of two functions $w(f, g)$ is defined as $w(f, g) := \begin{vmatrix} f & g \\ D_xf & D_xg \end{vmatrix}$

Lemma: If $w(f, g) \equiv 0$ $(w(f, g)(x) = 0, \forall x)$, then f, g are linearally dependent. If $w(f, g) \not\equiv 0$ $(\exists x$ s.t. $w(f, g)(x) \neq 0)$, then f, g are linearally independent.

Then $D_xu_1 = \frac{-y_2g}{w(y_1,y_2)}$ and $D_xu_2 = \frac{y_1g}{w(y_1,y_2)}$. This tells us that $y_p = u_1y_1 + u_2y_2$ (notice that u_1, u_2 are not derivatives).

Equivelently, this is:

$$\begin{aligned} y_p &= -y_1(x)\int_{x_0}^{x} \frac{y_2(s)g(s)}{w(y_1, y_2)(s)}ds + y_2(x)\int_{x_0}^{x} \frac{y_1(s)g(s)}{w(y_1, y_2)(s)}ds \\ &= \int_{x_0}^{x} \frac{y_2(x)y_1(s) - y_1(x)y_2(s)}{w(y_1, y_2)(s)} g(s)ds = 0 \end{aligned}$$

Where x_0 is a convenient point int the interval I in which y_1, y_2 are defined.

3 Nth Order

A nth order differential equation is (as the name implies), a differential equation that includes derrivatives of n orders.

3.1 Nth Order Constant Coefficient Homogeneous

A nth order constant coefficient homogeneous D.E. is one with the form $a_0 D_x^n y + a_1 D_x^{n-1} y + \cdots + a_{n-1} D_x y + a_n y = 0$.

To solve, by following the same procedure as for the 2nd order parallel, let $y = e^{rx} \Rightarrow a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$, and solve.

4 Systems of Differential Equations

4.1 First Order

Definition: A system of first order differential equations is defined as such: $D_t x_i = \sum_{j=1}^{n} (P_{ij}(t) x_j) + g_i(t)$ for $1 \leq i \leq n \Leftrightarrow D_t \vec{x} = \left[P_{ij}(t)\right]_{ij} \vec{x} = \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}$

A system of first order D.E.s is homogeneous if $g_i(t) = 0, \forall t$.

Theorem: If $\{\vec{e_i}\}$ is a standard basis for $\Re^n$, $\vec{x_i}$ is a solution to the homogeneous system with the initial condidtion $\vec{x_i}(t_0) = \vec{e_i}$ then $\{\vec{x_i}\}$ is the fundimental solution set.

4.1.1 Eigenvalue Method for Homogeneous Systems

For a homogeneous system, you have $D_x \vec{x} = A\vec{x}$. If you can find the eigenvalues $\lambda_1, \cdots, \lambda_n$ and eigenvectors $\vec{v_1}, \cdots, \vec{v_n}$, then $\vec{x} = \sum_{i=1}^{n} c_i e^{\lambda_i} \vec{v_i}$

If there are repeated eigenvalues, solve for the known one like normal: (In this example I'm just showing for a system of two, but it extends) $\vec{x} = \vec{x_1} + \vec{x_2}$, $\vec{x_1} = c_1 \vec{v_1} e^{\lambda_1 t}$, let $\vec{x_2} = \vec{w} t e^t \Rightarrow D_t \vec{x_2} = \vec{w}(e^t + te^t)$. Then plug back into initial D.E., and solve for $\vec{w}$.

4.1.2 Converting to and from systems

Given a second order, often you can convert to a system of first orders, or vice versa. This is done by assigning the variables in the system to be different level derivatives.

Example 4.1.2.1: $a D_x^2 x + bx = 0 \Rightarrow$ let $x_1 = x, x_2 = D_x x \Rightarrow D_x x_2 = \frac{-bx_1}{a}, D_x x_1 = x_2$

You can also go from multiple of a higher order to lower orders:

Example 4.1.2.2: For two second orders we define $y_1 = x_1$, $y_2 = D_x x_1$, $y_3 = x_2$, $y_4 = D_x x_2$

4.1.3 Matrix Exponents

Consider $D_t X = AX \; \forall X \in M_{n\times n}(S)$. If $\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + \cdots + c_n\vec{x_n}$ (means $\vec{x_i} = \vec{v_i}e^{\lambda_i t}$ most likely) is a solution to $D_t\vec{x} = A\vec{x}$, then there is a solution $X = \Phi(t) = \begin{bmatrix}\vec{x_1} & \vec{x_2} & \cdots & \vec{x_n}\end{bmatrix}$ where $\vec{x_i}$ are the columns of the matrix. If there is an initial condition $\vec{x}(0) = \vec{x_0}$ then $\Phi(t)\vec{c} = \vec{x_0} \Leftrightarrow \vec{c} = \Phi^{-1}(t)\vec{x}$

Then to solve $D_t X = AX$, $\vec{x}(0) = \vec{x_0}$, we have $\vec{x}(t) = \Phi(t)\Phi^{-1}(0)\vec{x_0}$

To solve $D_t X = AX$, we really want $X = e^{At}$. We know $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^A = \sum\limits_{n=0}^{\infty} \frac{A^n}{n!} \Rightarrow e^{At} = \sum\limits_{n=0}^{\infty} \frac{A^n t^n}{n!}$

Note 4.1.3.1: If $AB = BA$, $e^{A+B} = e^A e^B; (e^A)^{-1} = e^{-A}; e^{0\in M_{n\times n}(S)} = I$

If $A = diag(a_1, a_2, \cdots, a_n)$ then $e^A = diag(e^{a_1}, e^{a_2}, \cdots, e^{a_n})$
If $A = SDS^{-1}$ for a diagonal D then $e^A = Se^D S^{-1}$.
if A is non-diagonalizable, then cehck if there is n s.t. $A^n = 0$. If so, than a polynomial can be constructed from $e^A = \sum\limits_{i=0}^{n} \frac{A^i}{i!} \left(= \sum\limits_{n=0}^{\infty} \frac{A^n}{n!}\right)$

Example 4.1.3.2 For $A = \begin{bmatrix}0 & 1 & 2\\ 0 & 0 & 3\\ 0 & 0 & 0\end{bmatrix}$, $A^2 \neq 0$, $A^3 = 0 \Rightarrow e^A = \sum\limits_{i=0}^{2} \frac{A^i}{i!} = (A^0 = I) + A + \frac{1}{2}A^2$

Note 4.1.3.3: If $AB = BA$, $C = A + B$ then $e^{Ct} = e^{At}e^{Bt}$. Note that if $A = nI$, $AB = nIB = nB = Bn = BnI = BA$.

Thus the solution to $D_t\vec{x} = a\vec{x}$, $\vec{x}(0) = \vec{x_0}$ is $\vec{x}(t) = e^{At}\vec{x_0}$ $(= \Phi(t)\Phi^{-1}(0)\vec{x_0} \Rightarrow e^{At} = \Phi(t)\Phi^{-1}(0)\vec{x_0})$